CHAPTER 9

THIN CYLINDERS AND SHELLS

Summary

The stresses set up in the walls of a thin cylinder owing to an internal pressure $p$ are:

- circumferential or hoop stress $\sigma_h = \frac{pd}{2t}$
- longitudinal or axial stress $\sigma_L = \frac{pd}{4t}$

where $d$ is the internal diameter and $t$ is the wall thickness of the cylinder.

Then:

- longitudinal strain $\varepsilon_L = \frac{1}{E} [\sigma_L - \nu \sigma_h]$
- hoop strain $\varepsilon_h = \frac{1}{E} [\sigma_h - \nu \sigma_L]$

change of internal volume of cylinder under pressure $= \frac{pd}{4tE} [5 - 4\nu] V$

change of volume of contained liquid under pressure $= \frac{pV}{K}$

where $K$ is the bulk modulus of the liquid.

For thin rotating cylinders of mean radius $R$ the tensile hoop stress set up when rotating at $\omega$ rad/s is given by

$\sigma_h = \rho \omega^2 R^2$.

For thin spheres:

- circumferential or hoop stress $\sigma_h = \frac{pd}{4t}$
- change of volume under pressure $= \frac{3pd}{4tE} [1 - \nu] V$

Effects of end plates and joints—add “joint efficiency factor” $\eta$ to denominator of stress equations above.

9.1. Thin cylinders under internal pressure

When a thin-walled cylinder is subjected to internal pressure, three mutually perpendicular principal stresses will be set up in the cylinder material, namely the circumferential or hoop
stress, the radial stress and the longitudinal stress. Provided that the ratio of thickness to inside diameter of the cylinder is less than 1/20, it is reasonably accurate to assume that the hoop and longitudinal stresses are constant across the wall thickness and that the magnitude of the radial stress set up is so small in comparison with the hoop and longitudinal stresses that it can be neglected. This is obviously an approximation since, in practice, it will vary from zero at the outside surface to a value equal to the internal pressure at the inside surface. For the purpose of the initial derivation of stress formulae it is also assumed that the ends of the cylinder and any riveted joints present have no effect on the stresses produced; in practice they will have an effect and this will be discussed later (§9.6).

9.1.1. Hoop or circumferential stress

This is the stress which is set up in resisting the bursting effect of the applied pressure and can be most conveniently treated by considering the equilibrium of half of the cylinder as shown in Fig. 9.1.

![Fig. 9.1. Half of a thin cylinder subjected to internal pressure showing the hoop and longitudinal stresses acting on any element in the cylinder surface.](image)

Total force on half-cylinder owing to internal pressure \(= p \times \text{projected area} = p \times dL \)

Total resisting force owing to hoop stress \(\sigma_H\) set up in the cylinder walls

\[= 2\sigma_H \times Lt\]

\[\therefore 2\sigma_H Lt = pdL\]

\[\therefore \text{circumferential or hoop stress } \sigma_H = \frac{pd}{2t} \] (9.1)

9.1.2. Longitudinal stress

Consider now the cylinder shown in Fig. 9.2.

Total force on the end of the cylinder owing to internal pressure

\[= \text{pressure} \times \text{area} = p \times \frac{\pi d^2}{4}\]
Area of metal resisting this force = $\pi dt$ (approximately)

\[
\text{stress set up} = \frac{\text{force}}{\text{area}} = p \times \frac{\pi d^2/4}{\pi dt} = \frac{pd}{4t}
\]

i.e.

longitudinal stress $\sigma_L = \frac{pd}{4t}$  

(9.2)

9.1.3 Changes in dimensions

(a) Change in length

The change in length of the cylinder may be determined from the longitudinal strain, i.e. neglecting the radial stress.

Longitudinal strain $= \frac{1}{E} [\sigma_L - v\sigma_H]$

and

\[
\text{change in length} = \text{longitudinal strain} \times \text{original length} = \frac{1}{E} [\sigma_L - v\sigma_H] L
\]

\[
= \frac{pd}{4tE} [1 - 2v] L
\]

(9.3)

(b) Change in diameter

As above, the change in diameter may be determined from the strain on a diameter, i.e. the diametral strain.

Diametral strain $= \frac{\text{change in diameter}}{\text{original diameter}}$

Now the change in diameter may be found from a consideration of the circumferential change. The stress acting around a circumference is the hoop or circumferential stress $\sigma_H$ giving rise to the circumferential strain $\varepsilon_H$.

Change in circumference $= \text{strain} \times \text{original circumference} = \varepsilon_H \times \pi d$
§9.2 Thin Cylinders and Shells

New circumference = \( \pi d + \pi d \varepsilon_h \)
\[ = \pi d (1 + \varepsilon_h) \]

But this is the circumference of a circle of diameter \( d (1 + \varepsilon_h) \)

\[ \therefore \quad \text{New diameter} = d (1 + \varepsilon_h) \]

\[ \therefore \quad \text{Change in diameter} = d \varepsilon_h \]

Diametral strain \( \varepsilon_d = \frac{d \varepsilon_h}{d} = \varepsilon_h \)

i.e. the diametral strain equals the hoop or circumferential strain \( (9.4) \)

Thus change in diameter = \( d \varepsilon_h = \frac{d}{E} [\sigma_h - \nu \sigma_L] \)
\[ = \frac{pd^2}{4tE} [2 - \nu] \quad (9.5) \]

(c) Change in internal volume

Change in volume = volumetric strain \( \times \) original volume

From the work of §14.5, page 364.

volumetric strain = sum of three mutually perpendicular direct strains
\[ = \varepsilon_L + 2 \varepsilon_D \]
\[ = \frac{1}{E} [\sigma_L - \nu \sigma_H] + \frac{2}{E} [\sigma_H - \nu \sigma_L] \]
\[ = \frac{1}{E} [\sigma_L + 2 \sigma_H - \nu (\sigma_H + 2 \sigma_L)] \]
\[ = \frac{pd}{4tE} [1 + 4 - \nu (2 + 2)] \]
\[ = \frac{pd}{4tE} [5 - 4\nu] \]

Therefore with original internal volume \( V \)

change in internal volume = \( \frac{pd}{4tE} [5 - 4\nu] V \) \( (9.6) \)

9.2. Thin rotating ring or cylinder

Consider a thin ring or cylinder as shown in Fig. 9.3 subjected to a radial pressure \( p \) caused by the centrifugal effect of its own mass when rotating. The centrifugal effect on a unit length
of the circumference is:

\[ p = m \omega^2 r \]

Thus, considering the equilibrium of half the ring shown in the figure:

\[ 2F = p \times 2r \]

\[ F = pr \]

where \( F \) is the hoop tension set up owing to rotation.

The cylinder wall is assumed to be so thin that the centrifugal effect can be taken to be constant across the wall thickness.

\[ F = pr = m \omega^2 r^2 \]

This tension is transmitted through the complete circumference and therefore is restricted by the complete cross-sectional area.

\[ \therefore \text{hoop stress} = \frac{F}{A} = \frac{m \omega^2 r^2}{A} \]

where \( A \) is the cross-sectional area of the ring.

Now with unit length assumed, \( m/A \) is the mass of the ring material per unit volume, i.e. the density \( \rho \).

\[ \therefore \text{hoop stress} = \rho \omega^2 r^2 \quad (9.7) \]

### 9.3. Thin spherical shell under internal pressure

Because of the symmetry of the sphere the stresses set up owing to internal pressure will be two mutually perpendicular hoop or circumferential stresses of equal value and a radial stress. As with thin cylinders having thickness to diameter ratios less than 1:20, the radial stress is assumed negligible in comparison with the values of hoop stress set up. The stress system is therefore one of equal biaxial hoop stresses.

Consider, therefore, the equilibrium of the half-sphere shown in Fig. 9.4.

Force on half-sphere owing to internal pressure

\[ = \text{pressure} \times \text{projected area} \]

\[ = p \times \frac{\pi d^2}{4} \]

Resisting force \( = \sigma_r \times \pi dt \) (approximately)
Thin Cylinders and Shells

Fig. 9.4. Half of a thin sphere subjected to internal pressure showing uniform hoop stresses acting on a surface element.

\[ p \times \frac{\pi d^2}{4} = \sigma_H \times \pi dt \]

or

\[ \sigma_H = \frac{pd}{4t} \]

i.e. \textbf{circumferential or hoop stress} = \(\frac{pd}{4t}\) \hspace{1cm} (9.8)

\section*{9.3.1. Change in internal volume}

As for the cylinder,

\[ \text{change in volume} = \text{original volume} \times \text{volumetric strain} \]

but\[\text{volumetric strain} = \text{sum of three mutually perpendicular strains (in this case all equal)} \]

\[ = 3\varepsilon_D = 3\varepsilon_H \]

\[ = \frac{3}{E} [\sigma_H - \nu \sigma_H] \]

\[ = \frac{3pd}{4tE} [1 - \nu] \]

\[ \therefore \text{change in internal volume} = \frac{3pd}{4tE} [1 - \nu] V \]

\hspace{1cm} (9.9)

\section*{9.4. Vessels subjected to fluid pressure}

If a fluid is used as the pressurisation medium the fluid itself will change in volume as pressure is increased and this must be taken into account when calculating the amount of fluid which must be pumped into the cylinder in order to raise the pressure by a specified amount, the cylinder being initially full of fluid at atmospheric pressure.

Now the \textit{bulk modulus} of a fluid is defined as follows:

\[ \text{bulk modulus } K = \frac{\text{volumetric stress}}{\text{volumetric strain}} \]
where, in this case, volumetric stress = pressure $p$

and

$$\text{volumetric strain} = \frac{\text{change in volume}}{\text{original volume}} = \frac{\delta V}{V}$$

\[ K = \frac{p}{\delta V/V} = \frac{pV}{\delta V} \]

i.e.

$$\text{change in volume of fluid under pressure} = \frac{pV}{K} \quad (9.10)$$

The extra fluid required to raise the pressure must, therefore, take up this volume together with the increase in internal volume of the cylinder itself.

\[ \text{extra fluid required to raise cylinder pressure by } p = \frac{pd}{4tE} [5 - 4v] V + \frac{pV}{K} \quad (9.11) \]

Similarly, for spheres, the extra fluid required is

$$= \frac{3pd}{4tE} [1 - v] V + \frac{pV}{K} \quad (9.12)$$

9.5. Cylindrical vessel with hemispherical ends

Consider now the vessel shown in Fig. 9.5 in which the wall thickness of the cylindrical and hemispherical portions may be different (this is sometimes necessary since the hoop stress in the cylinder is twice that in a sphere of the same radius and wall thickness). For the purpose of the calculation the internal diameter of both portions is assumed equal.

From the preceding sections the following formulae are known to apply:

Fig. 9.5. Cross-section of a thin cylinder with hemispherical ends.

(a) For the cylindrical portion

$$\text{hoop or circumferential stress} = \sigma_{hc} = \frac{pd}{2t_c}$$
Thin Cylinders and Shells

\[ \sigma_L = \frac{pd}{4t_c} \]

\[ \therefore \quad \text{hoop or circumferential strain} = \frac{1}{E} \left[ \sigma_H - v\sigma_L \right] \]

\[ = \frac{pd}{4t_cE} \left[ 2 - v \right] \]

(b) For the hemispherical ends

\[ \sigma_H = \frac{pd}{4t_s} \]

\[ \therefore \quad \text{hoop strain} = \frac{1}{E} \left[ \sigma_H - v\sigma_H \right] \]

\[ = \frac{pd}{4t_sE} \left[ 1 - v \right] \]

Thus equating the two strains in order that there shall be no distortion of the junction,

\[ \frac{pd}{4t_sE} \left[ 2 - v \right] = \frac{pd}{4t_cE} \left[ 1 - v \right] \]

i.e.

\[ \frac{t_s}{t_c} = \frac{(1 - v)}{(2 - v)} \quad (9.13) \]

With the normally accepted value of Poisson's ratio for general steel work of 0.3, the thickness ratio becomes

\[ \frac{t_s}{t_c} = \frac{0.7}{1.7} \]

i.e. the thickness of the cylinder walls must be approximately 2.4 times that of the hemispherical ends for no distortion of the junction to occur. In these circumstances, because of the reduced wall thickness of the ends, the maximum stress will occur in the ends. For equal maximum stresses in the two portions the thickness of the cylinder walls must be twice that in the ends but some distortion at the junction will then occur.

9.6. Effects of end plates and joints

The preceding sections have all assumed uniform material properties throughout the components and have neglected the effects of endplates and joints which are necessary requirements for their production. In general, the strength of the components will be reduced by the presence of, for example, riveted joints, and this should be taken into account by the introduction of a joint efficiency factor \( \eta \) into the equations previously derived.
Thus, for thin cylinders:

$$\text{hoop stress} = \frac{pd}{2\eta_L}$$

where \( \eta_L \) is the efficiency of the longitudinal joints,

$$\text{longitudinal stress} = \frac{pd}{4\eta_C}$$

where \( \eta_C \) is the efficiency of the circumferential joints.

For thin spheres:

$$\text{hoop stress} = \frac{pd}{4\eta}$$

Normally the joint efficiency is stated in percentage form and this must be converted into equivalent decimal form before substitution into the above equations.

### 9.7. Wire-wound thin cylinders

In order to increase the ability of thin cylinders to withstand high internal pressures without excessive increases in wall thickness, and hence weight and associated material cost, they are sometimes wound with high tensile steel tape or wire under tension. This subjects the cylinder to an initial hoop, compressive, stress which must be overcome by the stresses owing to internal pressure before the material is subjected to tension. There then remains at this stage the normal pressure capacity of the cylinder before the maximum allowable stress in the cylinder is exceeded.

It is normally required to determine the tension necessary in the tape during winding in order to ensure that the maximum hoop stress in the cylinder will not exceed a certain value when the internal pressure is applied.

Consider, therefore, the half-cylinder of Fig. 9.6, where \( \sigma_H \) denotes the hoop stress in the cylinder walls and \( \sigma_t \) the stress in the rectangular-sectioned tape. Let conditions before pressure is applied be denoted by suffix 1 and after pressure is applied by suffix 2.

![Fig. 9.6. Section of a thin cylinder with an external layer of tape wound on with a tension.](image-url)