Leaf springs

Characteristics

1. Sometimes it is also called as a semi-elliptical spring; as it takes the form of a slender arc shaped length of spring steel of rectangular cross section.

2. The center of the arc provides the location for the axle, while the tie holes are provided at either end for attaching to the vehicle body.

3. Supports the chassis weight

4. Controls chassis roll more efficiently-high rear moment center and wide spring base

5. Controls rear end wrap-up

6. Controls axle damping

7. Controls braking forces

8. Regulates wheelbase lengths (rear steer) under acceleration and braking

Leaf Springs

In the cantilever beam type leaf spring, for the same leaf thickness, h, leaf of uniform width, b (case 1) and, leaf of width, which is uniformly reducing from b (case 2) is considered. From the basic equations of bending stress and deflection, the maximum stress $\sigma_{\text{max}}$, and tip deflection $\delta_{\text{max}}$, can be derived.
It is observed that instead of uniform width leaf, if a leaf of varying width is used, the bending stress at any cross section is same and equal to maximum stress $\sigma_{\text{max}}$. This is called as leaf of a uniform strength.

Moreover, the tip deflection being more, comparatively, it has greater resilience than its uniform width counterpart.

Resilience, as we know, is the capacity to absorb potential energy during deformation.

**For case 1 (uniform width)**

$$\sigma_{\text{max}} = \frac{6FL}{bh^3}$$

$$\delta_{\text{max}} = \frac{4FL^3}{Eb^3}$$

**For case 2 (non uniform width)**

$$\sigma_{\text{max}} = \frac{6FL}{bh^3}$$

$$\delta_{\text{max}} = \frac{6FL^3}{Eb^3}$$

**For case 1 (uniform width)**

$$\sigma_{\text{max}} = \frac{3FL}{bh^2}$$

$$\delta_{\text{max}} = \frac{2FL^3}{Eb^3}$$

$$\sigma_{\text{max}} = \frac{3FL}{bh^2}$$

$$\delta_{\text{max}} = \frac{3FL^3}{Eb^3}$$
One of the applications of leaf spring of simply supported beam type is seen in automobiles, where, the central location of the spring is fixed to the wheel axle. Therefore, the wheel exerts the force $F$ on the spring and support reactions at the two ends of the spring come from the carriage.

**Design theme of a leaf spring**

Let us consider the simply supported leaf of Lozenge shape for which the maximum stress and maximum deflection are known.

From the stress and deflection equations the thickness of the spring plate, $h$, can be obtained as,

$$h = \frac{\sigma_{\text{max}} L^2}{E \delta_{\text{max}}} = \frac{\sigma_{\text{des}} L^2}{E \delta_{\text{des}}}$$

The $\sigma_{\text{max}}$ is replaced by design stress $\sigma_{\text{des}}$ similarly, $\delta_{\text{max}}$ is replaced by $\delta_{\text{des}}$. $E$ is the material property and depends on the type of spring material chosen.

$L$ is the characteristic length of the spring.

Therefore, once the design parameters, given on the left side of the above equation, are fixed the value of plate thickness, $h$ can be calculated.

Substitution of $h$ in the stress equation above will yield the value of plate width $b$.

$$b = \frac{31' L}{\sigma_{\text{des}} h^2}$$

In the similar manner $h$ and $b$ can be calculated for leaf springs of different support conditions and beam types.

**Laminated springs**
One of the difficulties of the uniform strength beam, say Lozenge shape, is that the value of width $b$ sometimes is too large to accommodate in a machine assembly. One practice is that instead of keeping this large width one can make several slices and put the pieces together as a laminate. This is the concept of laminated spring. The Lozenge shaped plate is cut into several longitudinal strips, as indicated in the figure.

The central strip, marked 1 is the master leaf which is placed at the top. Then two pieces, marked 2 are put together, side by side to form another leaf and placed below the top leaf. In the similar manner other pairs of strips, marked 3 and 4 respectively are placed in the decreasing order of strip length to form a laminated spring. Here width of each strip, $b_N$ is given as;

$$b_N = \frac{b}{N} \quad \text{or} \quad b' = \frac{b}{i}$$

Where $N$ is the number of strips

The stress and deflection equations for a laminated spring is,

$$\sigma_{max} = \frac{C_1 F L}{i b' h^2} \quad \text{and} \quad \delta_{max} = \frac{C_2 F L^3}{E i b' h^3}$$

Where, constants $C_1$ and $C_2$ are different for different cases,

The values of the constants $C_1$ and $C_2$ for cantilever beam case
### Cantilever Beam Constants

<table>
<thead>
<tr>
<th></th>
<th></th>
<th>$C_1$</th>
<th>$C_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Uniform Width</strong></td>
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<td>6</td>
<td>4</td>
</tr>
<tr>
<td><strong>Non-Uniform Width</strong></td>
<td><img src="image2" alt="Diagram" /></td>
<td>6</td>
<td>6</td>
</tr>
</tbody>
</table>

The values of the constants $C_1$ and $C_2$ for simply supported beam case

### Simply Supported Beam Constants

<table>
<thead>
<tr>
<th></th>
<th></th>
<th>$C_1$</th>
<th>$C_2$</th>
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</thead>
<tbody>
<tr>
<td><strong>Uniform Width</strong></td>
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<td>2</td>
</tr>
<tr>
<td><strong>Non-Uniform Width</strong></td>
<td><img src="image4" alt="Diagram" /></td>
<td>3</td>
<td>3</td>
</tr>
</tbody>
</table>
Laminated semi-elliptic spring

The figure shows a laminated semi-elliptic spring. The top leaf is known as the master leaf. The eye is provided for attaching the spring with another machine member. The amount of bend that is given to the spring from the central line, passing through the eyes, is known as camber. The camber is provided so that even at the maximum load the deflected spring should not touch the machine member to which it is attached. The central clamp is required to hold the leaves of the spring.

To prove that stress developed in the full length leaves is 50% more than that in the graduated leaves.

Step1: Bending stress and displacement in the graduated leaves

For analysis half the spring can be considered as a cantilever. It is assumed that the individual leaves are separated and the master leaf placed at the center. Then the second leaf is cut longitudinally into two halves, each of width (b/2) and placed on each side of the master leaf. A similar procedure is repeated for rest of the leaves.

The graduated leaves along with the master leaf thus can be treated as a triangular plate of thickness ‘t’ as shown in figure 1.
Let,

- $i_f = \text{No. of extra full length leaves}$
- $i_g = \text{No. of graduated leaves including the master leaf}$
- $b = \text{Width of each leaf}$
- $t = \text{Thickness of each leaf}$
- $L = \text{Length of the cantilever or half the length of the spring}$
- $F = \text{Total force applied at the end of the spring}$
- $F_f = \text{Force absorbed by the full length leaves.}$
- $F_g = \text{Force absorbed by the graduated leaves.}$

The bending stress developed in the graduated leaves will be:

$$
\sigma_{gy} = \frac{M_{gy}Y}{l} = \frac{F_g L}{12 (t_y b) t^3} = \frac{6 F_g L}{i_g b t^2}
$$
For cantilever triangular plate, the deflection at the point of application of force is given by:

\[ \delta_g = \frac{F_g L^3}{2EI_{\text{max}}} = \frac{6F_g L^3}{Ei_g b t^3} \]

**Step 2: Bending stress and displacements in full length leaves**

It is assumed that the individual leaves are separated and the full length leaf is placed at the center. Then the second full length leaf is cut longitudinally into two halves, each of width \( \frac{b}{2} \) and placed on each side of the first leaf. A similar procedure is repeated for the rest of the leaves.

The resulting cantilever beam of thickness ‘t’ is shown in the figure 2.

![Fig 2](image)

The bending stress developed in the full length leaves will be:

\[ \sigma_{bf} = \frac{M_{bf}y}{I} = \frac{F_f L \left( \frac{t}{2} \right)}{\frac{1}{12}(t_t b) t^3} = \frac{6F_f L}{i_f b t^2} \]

For a cantilever rectangular plate, the deflection at the point of application of force is given by:

\[ \delta_f = \frac{F_f L^3}{3EI_{\text{max}}} = \frac{4F_f L^3}{Ei_f b t^3} \]

**Step 3:**

\[ \delta = \delta_g = \delta_f \]
Since the graduated leaves and the full leaves are clamped together the deflection for both should be the same.

\[ \delta = \delta_g = \delta_f \]

\[ \frac{6F_gL^3}{EI_gbt^3} = \frac{4F_fL^3}{EI_fbt^3} \]

\[ \frac{F_g}{F_f} = \frac{2l_f}{3l_f} \]

Also

\[ F = F_g + F_f \]

\[ F = F_f \left( 1 + \frac{2l_g}{3l_f} \right) \]

\[ \frac{F_f}{F_f} = \frac{2l_g}{3l_f} \]

Solving we get;

\[ F_f = \left( \frac{3F_l}{3l_f + 2l_g} \right) \quad \text{And} \quad \frac{F_g}{F_g} = \left( \frac{2F_l}{3l_f + 2l_g} \right) \]

Substituting the values of \( F_f \) and \( F_g \) in the equations of \( \sigma_{bf} \) and \( \sigma_{bg} \) we get;

\[ \sigma_{bf} = \frac{12FL}{(3l_f + 2l_g)bt^2} \quad \text{And} \quad \sigma_{bg} = \frac{12FL}{(3l_f + 2l_g)bt^2} \]

Taking the ratios of both stresses and solving we get;

\[ \frac{\sigma_{bf}}{\sigma_{bg}} = 1.5 \]

Hence proved.
The maximum deflection of the leaf spring can be found as follows;

We have from previous equations

\[ F_f = \left( \frac{3F_i f}{3i_f + 2i_g} \right); \quad F_g = \left( \frac{2F_i g}{3i_f + 2i_g} \right) \]

And

\[ \delta_g = \frac{6F_g L^3}{Ei_g bt^3}; \quad \delta_f = \frac{4F_f L^3}{Ei_f bt^3} \]

Consider anyone of the deflection equation

i.e.; \[ \delta_f = \frac{4F_f L^3}{Ei_f bt^3} \]

Substituting \[ F_f = \left( \frac{3F_i f}{3i_f + 2i_g} \right) \]

in the above equation and solving, we get

Maximum deflection

\[ \delta = \frac{12FL^3}{(3i_f + 2i_g)Ebt^3} \]

Equalized stress in spring leaves (Nipping);
The stress in the full length leaves is 50% greater than the stress in the graduated leaves.

To distribute this additional stress from the full length leaves, pre-stressing is done. This is achieved by bending the leaves to different radii of curvature, before they are assembled with the centre bolt.

The full length leaves are given in greater radii of curvature than the adjacent one. Due to the different radii of curvature, when the full length leaves are staked with the graduated leaves, without bolting, a gap is observed between them. This gap is called Nip.

The nip eliminated by tightening of the center bolt due to these pre-stresses is induced in the leaves. This method of pre-stressing by giving different radii of curvature is called as nipping.

By giving a greater radius of curvature to the full length leaves than graduated leaves before the leaves are assembled to form a spring.

**Nip: C**

The value of the initial Nip C is nothing but the difference in deflection between the full length and the graduated leaves

\[ C = \delta_g - \delta_f \]

Solving, we get;

\[ C = \frac{2FL^3}{ib^2h^3E} \]

**Problem 20**

Determine the width and thickness of a flat spring carrying a central load of 5000N. The deflection is limited to 100mm. The spring is supported at both ends at a distance of 800mm. The allowable stress is 300N/mm² and modulus of elasticity 221GPa. The spring is of constant thickness and varying width.
Given data:

\[ F = 5000 \text{N}; \quad y = 100 \text{mm}; \]
\[ 2l = 800 \text{mm} \quad \therefore l = 400 \text{mm} \]
\[ \sigma = 300 \text{Nmm}^2; \quad E = 221 \text{GPa} = 221 \times 10^3 \text{N/mm}^2 \]

Solution:

Since the spring is of constant thickness and varying width. It is as shown in figure and from table

\[ c_1 = 3; \quad c_2 = 3 \]

Maximum stress in the spring
\[ \sigma = \frac{c_1 Fl}{bh^2} \]

i.e.

\[ 300 = \frac{3 \times 5000 \times 400}{bh^2} \]

\[ \therefore bh^2 = 20000 \quad \text{......... (1)} \]

Maximum deflection
\[ y = \frac{C_2 Fl^3}{Ebh^3} \]

\[ \therefore 100 = \frac{3 \times 5000 \times 400^3}{221 \times 10^3 bh^3} \]
\[ bh^3 = 43438.914 \] ............ (2)

eqn. (2) divided by eqn. (1)

\[
\frac{bh^3}{bh^2} = h = \frac{43438.914}{20000} = 2.172
\]

Take thickness of spring \( h = 2.5 \text{mm} \)

Width of spring at the centre,

From equation (1)

\[
b = \frac{20000}{2.5^2} = 3200 \text{mm}
\]

From equation (2)

\[
b = \frac{43438.914}{2.5^3} = 2780 \text{mm}
\]

Select the bigger value as the permissible value

\[ \therefore b = 3200 \text{mm} \]

The values of the constants \( C_1 \) and \( C_2 \) for cantilever beam case

<table>
<thead>
<tr>
<th>Cantilever Beam</th>
<th>Constants</th>
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<td>6</td>
</tr>
<tr>
<td>Non-Uniform Width</td>
<td>6</td>
</tr>
</tbody>
</table>
Problem 21

An automobile semi-elliptical leaf spring has 12 numbers of graduated leaves and 3 number of full length leaves. The spring is to sustain a load of 25kN at its center and the ratio of total depth to the width of the spring is 2.5. The material of the leaves has design normal stress of 450 MPa and a modulus of elasticity of 207 GPa. Determine

1. Width and thickness of leaves.
2. Initial gap between the full length and graduated leaves before assembly.
3. Bolt load
4. Central deflection.
5. Radius of curvature of first full length leaf.
6. The width of the central band is 100mm and the span of the leaves is 1200mm.

Given data;

- $i_g = 12$,
- $\sigma_f = 450\text{MPa}$
- $i_f = 3$
- $2F = 25\text{kN}$
- $E = 207\text{GPa}$
- $L_b = 100\text{mm}$
- $2L = 1200\text{mm}$

Solution:

Effective length

$$i = \frac{2L - L_b}{2} = \frac{2(600) - 100}{2} = 550\text{mm}$$

$$\frac{ih}{b} = 2.5$$

$i = i_g + i_f$

$i = 12 + 3 = 15$
\[ b' = 6h \]

The maximum stress in the spring with the full length leaf pre-stress

\[ \sigma_f = \frac{3FL}{ib'h^2} \]

\[ 450 = \frac{3 \times 12500 \times 550}{15 \times 6 \times h^3} \]

\[ h = 7.98 \text{mm} \approx 8 \text{mm (std)} \]

\[ b' = 6 \times h = 6 \times 8 = 48 \text{mm} \]

The initial gap between the full length and graduated length

\[ c = 27.25 \text{mm} \]

The load on the clip bolts

\[ F_b = \frac{i_g t f F}{i(2i_g + 3i_f)} \]

\[ F_b = \frac{12 \times 3 \times 12500}{15(2 \times 12 + 3 \times 3)} \]

\[ F_b = 909.1 \text{MPa} \]

Deflection of the spring

\[ y = \frac{6Ft^3}{bh^3E(2t_b + 3t_f)} \]

\[ y = \frac{6 \times 12500 \times 550^3}{48 \times 0^3 \times 207(2 \times 12 + 3 \times 3)} = 74.33 \text{mm} \]

Problem 22

A semi elliptical is to sustain a load of 25kN. The span of the spring is 1100mm with a central band of 100mm. The material selected for the leaves as a design normal stress of 400N/mm² and \( E = 207 \text{GPa} \). The ratio between total depth of the spring and width ‘2’ also determine the radius
of curvature to which the first full length leaf is to bend such that the spring becomes flat with the full load

Solution

$i_e = 10$, $\sigma_r = 400\text{MPa}$, $i_f = 2$

$2F = 25\text{kN}$, $E = 207\text{GPa}$, $L_b = 100\text{mm}$, $2L = 1100\text{mm}$

$$\frac{ih}{b'} = 2$$

Effective length

$$l = \frac{2L - L_b}{2}$$

$$\frac{2(550) - 100}{2} = 500\text{mm}$$

$$i = i_e + i_f$$

$$i = 10 + 2 = 12$$

$$ih/b' = 2$$

The maximum stress in the spring with the full length leaf pre-stress

$$\sigma_f = \frac{3FL}{ib'h^2}$$

$$\frac{12xh}{b'} = 2$$

$$b' = 6h$$

$$400 = \frac{3 \times 12500 \times 500}{12 \times 6xh^3}$$

$$h = 8.65\text{mm} \approx 10\text{mm} \text{ (std)}$$

$$b' = 6xh = 6 \times 10 = 60\text{mm}$$

The load on the clip bolts
\[ F_b = \frac{l_i l_f F}{l(2i_g + 3i_f)} \]

\[ F_b = \frac{2 \times 10 \times 12500}{12(2 	imes 10 + 3 	imes 2)} \]

\[ F_b = 801.2 \text{N} \]

Deflection of the spring

\[ \gamma = \frac{6Fl^3}{b'h^3E(2i_g + 3i_f)} \]

\[ \gamma = \frac{6 \times 12500 \times 500^3}{60 \times 10^3 \times 207(2 \times 10 + 3 \times 2)} = 29.03 \text{mm} \]

Combination of springs

Problem 23

A 100mm outside diameter steel coil spring having 10 active coils of 12.5 diameter wire is in contact with a 600mm long steel cantilever spring having 5 graduated leaves 100mm wide and 10 mm thick as shown in figure.

i) What force “F” is gradually applied to the top of the coil spring will cause the cantilever to deflect by 50mm

ii) What is the bending stress in cantilever beam?

iii) What is the shear stress in coil spring?

iv) What energies stored by each spring.

Take 210 GPa and G=84 GPa

Cantilever  
Coil  

\[ i = 5 \]

\[ i = 10 \]

\[ l = 600 \text{ mm} \]

\[ d = 12.5 \text{ mm} \]

\[ b' = 100 \text{ mm} \]

\[ D_0 = 100 \text{ mm} \]

\[ \therefore D = 100 - 12.5 = 87.5 \text{ mm} \]

\[ y = 50 \text{ mm} \]

\[ G = 84 \times 10^3 \text{ N/mm}^2 \]

\[ E = 210 \times 10^3 \text{ N/mm}^2 \]
\( h = 10 \text{ mm} \)

**Solution:**

**Spring Index**

\[
\c = \frac{D}{d} = \frac{87.5}{12.5} = 7
\]

Since the coil spring is on the top of the cantilever,

Load on cantilever = load on coil spring

i.e., \( F_1 = F_2 \)

For constant width varying depth

\( c_1 = 6 \) and \( c_2 = 8 \)

i) **Cantilever Spring**

**Deflection**

\[
y = \frac{c_2 F l^3}{E b' h^3}
\]

\[
\therefore \quad 50 = \frac{8 \times F_1 \times 600^3}{210 \times 10^8 \times 5 \times 100 \times 10^8}
\]

\( \therefore \) Load applied on the top of the coil spring

\( F_1 = 3038.2 \text{ N} \)

ii) **Bending Stress in cantilever spring**

\[
\sigma = \frac{c_1 F l}{l b' h^2} = \frac{6 \times 3038.2 \times 600}{5 \times 100 \times 10^2}
\]

\( \sigma = 218.75 \text{ N/mm}^2 \)

iii) **Shear Stress in coil spring**

\[
\tau = \frac{8 F D K}{\pi d^3}
\]
iv) Energy stored

Energy stored in the cantilever spring

\[ U_1 = \frac{1}{2} F_1 y_1 \]
\[ = \frac{1}{2} \times 3038.2 \times 50 \]
\[ = 75955 \text{ N-mm} \]
\[ U_2 \approx 76 \text{ Nm} \]

v) Energy stored in coil spring

\[ U_2 = \frac{1}{2} F_2 y_2 \]

\[ y_2 = \frac{8 F D^3}{G d^4} \]
\[ = \frac{8 \times 3038.2 \times 87.5^3 \times 10}{84 \times 10^4 \times 12.5^4} = 79.4 \text{ mm} \]
\[ \therefore U_2 = \frac{1}{2} \times 3038.2 \times 79.4 \]
\[ = 120618.54 \text{ Nmm} \]
\[ U_2 \approx 120.62 \text{ Nm} \]

Buckling of compression spring

Buckling is an instability that is normally shown up when a long bar or a column is applied with compressive type of load.

Similar situation arise if a spring is too slender and long then it sways sideways and the failure is known as buckling failure.

Buckling takes place for a compressive type of springs. Hence, the steps to be followed in design to avoid buckling are given below.
Free length (L) should be less than 4 times the coil diameter (D) to avoid buckling for most situations.

For slender springs central guide rod is necessary.

A guideline for free length (L) of a spring to avoid buckling is as follows,

\[ L < \frac{\pi D}{C_e} \sqrt{\frac{2(E - G)}{2G + E}} \]

\[ L < 2.57 \frac{D}{C_e}, \]  

For steel, where, \( c_e \) is the end condition and its value is given below

<table>
<thead>
<tr>
<th>( c_e )</th>
<th>End condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.0</td>
<td>Fixed and free end</td>
</tr>
<tr>
<td>1.0</td>
<td>hinged at both ends</td>
</tr>
<tr>
<td>0.707</td>
<td>hinged and fixed end</td>
</tr>
<tr>
<td>0.5</td>
<td>fixed at both ends</td>
</tr>
</tbody>
</table>

If the spring is placed between two rigid plates, then end condition may be taken as 0.5. If after calculation it is found that the spring is likely to buckle then one has to use a guide rod passing through the center of the spring axis along which the compression action of the spring takes place.

**Spring surge (critical frequency)**

If a load \( F \) act on a spring there is a downward movement of the spring and due to this movement a wave travels along the spring in downward direction and a to and fro motion continues.

This phenomenon can also be observed in closed water body where a disturbance moves toward the wall and then again returns back to the starting of the disturbance. This particular situation is called surge of spring.

If the frequency of surging becomes equal to the natural frequency of the spring the resonant frequency will occur which may cause failure of the spring.

Hence, one has to calculate natural frequency, known as the fundamental frequency of the spring and use a judgment to specify the operational frequency of the spring.
The fundamental frequency can be obtained from the relationship given below.

**Fundamental frequency:**

\[
f = \frac{1}{2} \sqrt{\frac{Kg}{W_s}} \quad \text{Both ends within flat plates}
\]

\[
f = \frac{1}{4} \sqrt{\frac{Kg}{W_s}} \quad \text{One end free and other end on flat}
\]

Where,  \( K \): Spring rate  
\( W_s \): Spring weight = \( 2.47\gamma d^2DN \)

Where \( K \) is the spring rate and \( W_s \) is the spring weight and \( d \) is the wire diameter, \( D \) is the coil diameter, \( N \) is the number of active coils and \( \gamma \) is the specific weight of spring material.

The operational frequency of the spring should be at least 15-20 times less than its fundamental frequency.

This will ensure that the spring surge will not occur and even other higher modes of frequency can also be taken care of.

**Questions and answers**

**What are the forms of leaf spring?**

Leaf springs are of two forms: cantilever and simply supported type.

**What does the term “uniform strength” in the context of leaf spring mean?**

If the leaf spring has a shape of uniformly varying width (say Lozenge shape) then the bending stress at all section remains uniform. The situation is also identical as before in case of varying thickness, the thickness should vary non-uniformly with length to make a beam of uniform strength (\( L/h^2 = \text{constant} \)). These leaves require lesser material; have more resilience compared to a constant width leaf. These types of springs are called leaf springs of uniform strength.

**What is “nipping” in a laminated spring?**

In general the differential curvature between the master leaf and the next leaves is provided in a laminated spring, where, radius of curvature being more for the master leaf. This construction reduces the stress in the master leaf as compared to the other leaves of the spring in a laminated spring. This type of constructional feature is termed as *nipping.*