Lecture-49  

Signal Space Dimensionality and Processing Gain

- Fundamental issue in SS systems is how much protection spreading can provide against interference.
- SS technique distribute low dimensional signal into large dimensional signal space (hide the signal).
- Jammer has only one option; to jam the entire space with fixed total power or to jam portion of signal space with large power.

Consider set of orthonormal basis functions:

\[
\varphi_k(t) = \begin{cases} 
\sqrt{2} \cos(2\pi f_c t) & kT_c \leq t \leq (k+1)T_c \\
0 & \text{otherwise} 
\end{cases} \\
\tilde{\varphi}_k(t) = \begin{cases} 
\sqrt{2} \sin(2\pi f_c t) & kT_c \leq t \leq (k+1)T_c \\
0 & \text{otherwise} 
\end{cases} \quad k = 0, 1, \ldots, N-1
\]

where

- \(T_c\) is chip duration,
- \(N\) is number of chips per bit.

Transmitted signal \(x(t)\) for the interval of an information bit is

\[
x(t) = c(t) s(t) = \pm \sqrt{\frac{2E_b}{T_b}} c(t) \cos(2\pi f_c t) = \pm \sqrt{\frac{E_b}{N}} \sum_{k=0}^{N-1} c_k \varphi_k(t) \quad 0 \leq t \leq T_b
\]
where, Eb is signal energy per bit.
PN Code sequence \{ c_0, c_1, \ldots, c_{N-1} \} with \( c_k = \pm 1 \)
Transmitted signal \( x(t) \) is therefore \( N \) dimensional and requires \( N \) orthonormal functions
to represent it.
n(t) represent interfering signal (jammer). As said jammer tries to places all its available
energy in exactly same \( N \) dimension signal space. But jammer has no knowledge of
signal phase. Hence tries to place equal energy in two phase coordinates that is cosine
and sine
As per that jammer can be represented as
\[
j(t) = \sum_{k=0}^{N-1} j_k \varphi_k(t) + \sum_{k=0}^{N-1} \tilde{j}_k \tilde{\varphi}_k(t) \quad 0 \leq t \leq T_b
\]
where
\[
j_k = \int_0^{T_b} j(t) \varphi_k(t) \, dt \quad k = 0, 1, \ldots, N - 1
\]
\[
\tilde{j}_k = \int_0^{T_b} j(t) \tilde{\varphi}_k(t) \, dt \quad k = 0, 1, \ldots, N - 1
\]
Thus \( j(t) \) is \( 2N \) dimensional, twice the dimension as that of \( x(t) \).
Average interference power of \( j(t) \)
\[
J = \frac{1}{T_b} \int_0^{T_b} j^2(t) \, dt
\]
\[
= \frac{1}{T_b} \sum_{k=0}^{N-1} j_k^2 + \frac{1}{T_b} \sum_{k=0}^{N-1} \tilde{j}_k^2
\]
as jammer places equal energy in two phase coordinates, hence
\[
\sum_{k=0}^{N-1} j_k^2 = \sum_{k=0}^{N-1} \tilde{j}_k^2
\]
\[
J = \frac{2}{T_b} \sum_{k=0}^{N-1} j_k^2
\]
To evaluate system performance we calculate SNR at input and output of DS/BPSK receiver.
The coherent receiver input is \( u(t) = s(t) + c(t)j(t) \) and using this \( u(t) \), output at coherent receiver

\[
v = \sqrt{\frac{2}{T_b}} \int_0^{T_b} u(t) \cos(2\pi f_c t) dt
\]

\[
= V_s + V_{cj}
\]

Where \( V_s \) is despread component of BPSK and \( V_{cj} \) of spread interference.

\[
V_s = \sqrt{\frac{2}{T_b}} \int_0^{T_b} s(t) \cos(2\pi f_c t) dt
\]

\[
V_{cj} = \sqrt{\frac{2}{T_b}} \int_0^{T_b} c(t) j(t) \cos(2\pi f_c t) dt
\]

Consider despread BPSK signal \( s(t) \)

\[
s(t) = \pm \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_c t) \quad 0 \leq t \leq T_b
\]

Where + sign is for symbol 1
- sign for symbol 0.
If carrier frequency is integer multiple of \( 1/T_b \), we have

\[
V_s = \pm \sqrt{E_b}
\]

Consider spread interference component \( V_{cj} \)
here \( c(t) \) is considered in sequence form \{ \( c_0, c_1, \ldots, c_{N-1} \) \}

\[
V_{cj} = \sqrt{\frac{T_c}{T_b}} \sum_{k=0}^{N-1} c_k \int_0^{T_b} j(t) \phi_k(t) dt
\]

\[
= \sqrt{\frac{T_c}{T_b}} \sum_{k=0}^{N-1} c_k j_k
\]

With \( c_k \) treated as independent identical random variables with both symbols having equal probabilities

\[
P(C_k = 1) = P(C_k = -1) = \frac{1}{2}
\]
Expected value of Random variable $v_{cj}$ is zero, for fixed $k$ we have

$$E[C_{kj}|j_k] = j_k P(C_k = 1) - j_k P(C_k = -1)$$
$$= \frac{1}{2} j_k - \frac{1}{2} j_k$$
$$= 0$$

and Variance

$$Var[V_{cj}|j] = \frac{1}{N} \sum_{k=0}^{N-1} j_k^2 = \frac{JT_c}{2}$$

Spread factor $N = T_b/T_c$

Output signal to noise ratio is

$$(SNR)_0 = \frac{2E_b}{JT_c}$$

The average signal power at receiver input is $E_b/T_b$, hence input SNR

$$(SNR)_i = \frac{E_b/T_b}{J}$$

$$(SNR)_0 = \frac{2T_b}{T_c}(SNR)_i$$

Expressing SNR in decibels

$$10\log_{10}(SNR)_0 = 10\log_{10}(SNR)_i + 3 + 10\log_{10}(PG), \text{dB}$$

where

$$PG = \frac{T_b}{T_c}$$

- .3db term on right side accounts for gain in SNR due to coherent detection.
- . Last term accounts for gain in SNR by use of spread spectrum.

$PG$ is called Processing Gain
1. Bit rate of binary data entering the transmitter input is
   \[ R_b = \frac{1}{T_b} \]

2. The bandwidth of PN sequence \( c(t) \) of main lobe is \( W_c \)
   \[ W_c = \frac{1}{T_c} \]

   Processing Gain, \( PG = \frac{W_c}{R_b} = \frac{R_c}{R_b} = \frac{T_c}{T_b} \)
**Probability of error**

To calculate probability of error, we consider output component $v$ of coherent detector as sample value of random variable $V$

$$V = \pm \sqrt{E_b} + V_{cj}$$

$E_b$ is signal energy per bit and $V_{cj}$ is noise component.

Decision rule is, if detector output exceeds a threshold of zero volts; received bit is symbol 1 else decision is favored for zero.

- Average probability of error $P_e$ is nothing but conditional probability which depends on random variable $V_{cj}$.
As a result receiver makes decision in favor of symbol 1 when symbol 0 transmitted and vice versa.

Random variable $V_{cj}$ is sum of $N$ such random variables. Hence for large $N$ it can assume Gaussian distribution.

As mean and variance has already been discussed, zero mean and variance $JT_c/2$.

Probability of error can be calculated from simple formula for DS/BPSK system

$$P_e = \frac{1}{2} \text{erfc} \left( \frac{E_b}{\sqrt{JT_c}} \right)$$

**Antijam Characteristics**

Consider error probability of BPSK

$$P_e = \frac{1}{2} \text{erfc} \left( \frac{E_b}{\sqrt{N_0}} \right)$$

Comparing both probabilities:

$$\frac{N_0}{2} = \frac{JT_c}{2}$$

Since bit energy $E_b = PT_b$, $P$ average signal power.

We can express bit energy to noise density ratio as

$$\frac{E_b}{N_0} = \left( \frac{T_b}{T_c} \right) \left( \frac{P}{J} \right)$$

or

$$\frac{J}{P} = \frac{PG}{E_b / N_0}$$

The ratio $J/P$ is termed jamming margin. Jamming Margin is expressed in decibels as

$$(\text{jamming margin})_{dB} = (\text{Processing gain})_{dB} - 10 \log_{10} \left( \frac{E_b}{N_0} \right)_{\text{min}}$$

Where $\left( \frac{E_b}{N_0} \right)_{\text{min}}$ is minimum bit energy to noise ratio needed to support a prescribed average probability of error.